

ECE 487
Lecture 7
Practical Frequency-Selective Digital Filter
Design

Dr. Emre Yengel

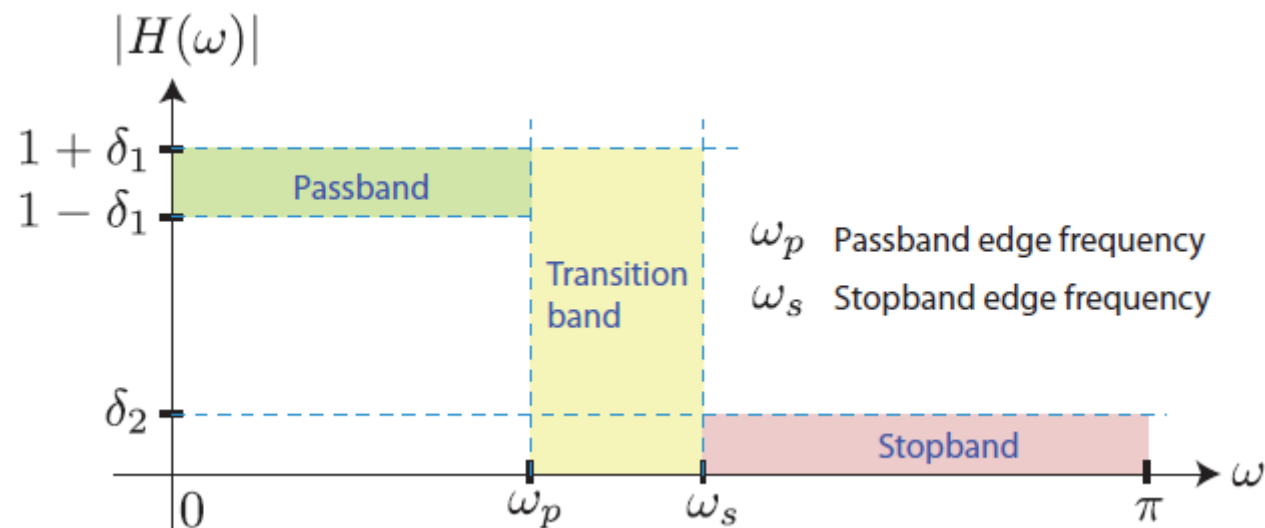
Cankaya University

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Source: ECEN448 Notes, Fall 2010, Texas A&M University

Digital Filter Design

- Desired filter characteristics are specified in the frequency domain in terms of desired magnitude and phase response of the filter; i.e., $H(\omega)$ is specified.



- Filter design involves determining the coefficients of a causal FIR or IIR filter that closely approximates the desired frequency response specifications.

FIR versus IIR Filters

- **FIR filters:** normally used when there is a requirement of **linear phase**
 - FIR filter with the following symmetry is linear phase:

$$h(n) = \pm h(M - 1 - n) \quad n = 0, 1, 2, \dots, M - 1$$

- **IIR filters:** normally used when linear phase is not required and **cost effectiveness** is needed
 - IIR filter has lower sidelobes in the stopband than an FIR having the same number of parameters
 - if some phase distortion is tolerable, an IIR filter has an implementation with fewer parameters requiring less memory and lower complexity:

Linear Phase

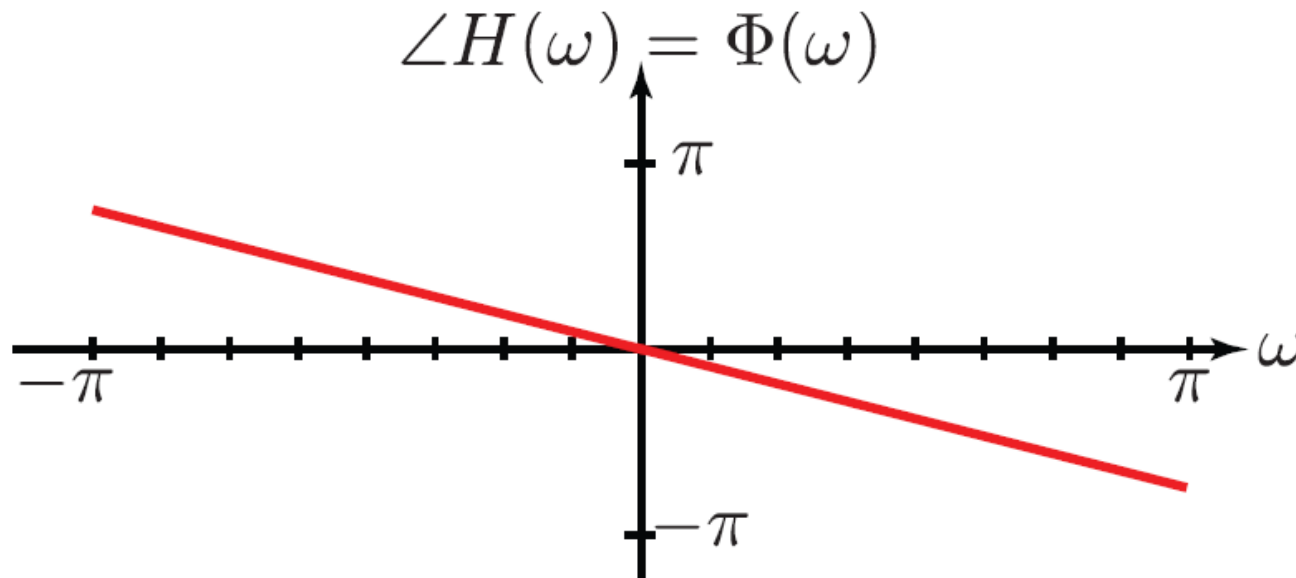
Q: What is linear phase?

A: The phase is a **straight line** in the **passband** of the system.

Linear Phase

Example: linear phase (all pass system)

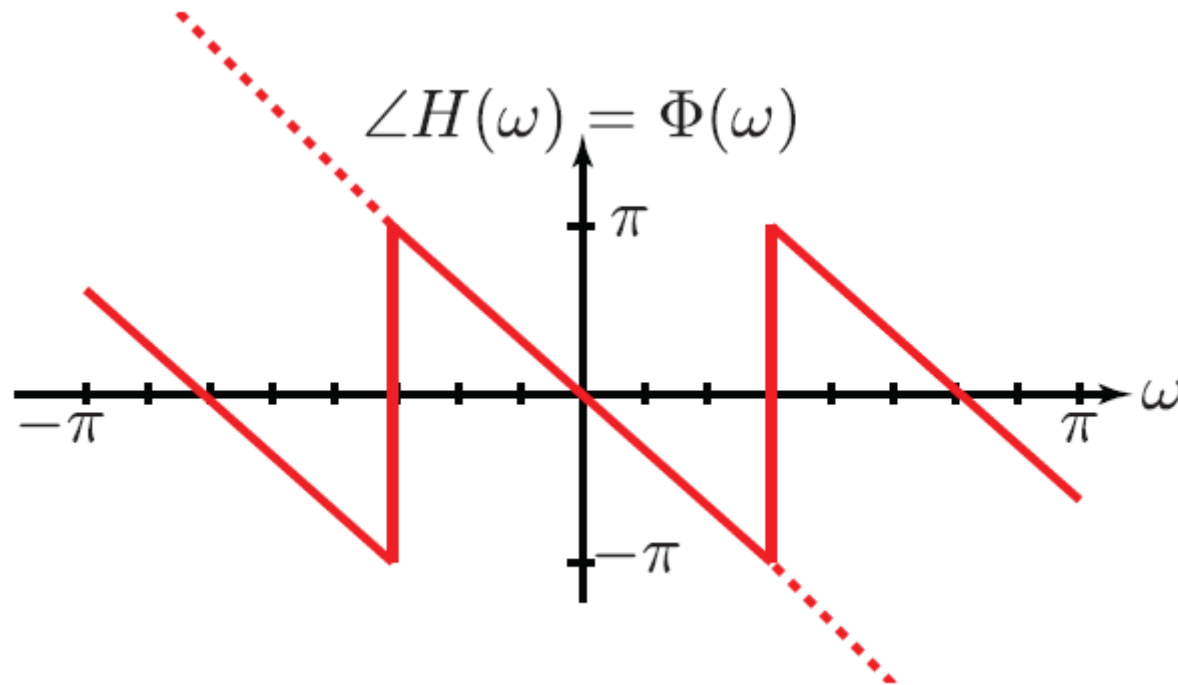
- **Group delay** is given by the negative of the slope of the line (more on this soon).



Linear Phase

Example: linear phase (all pass system)

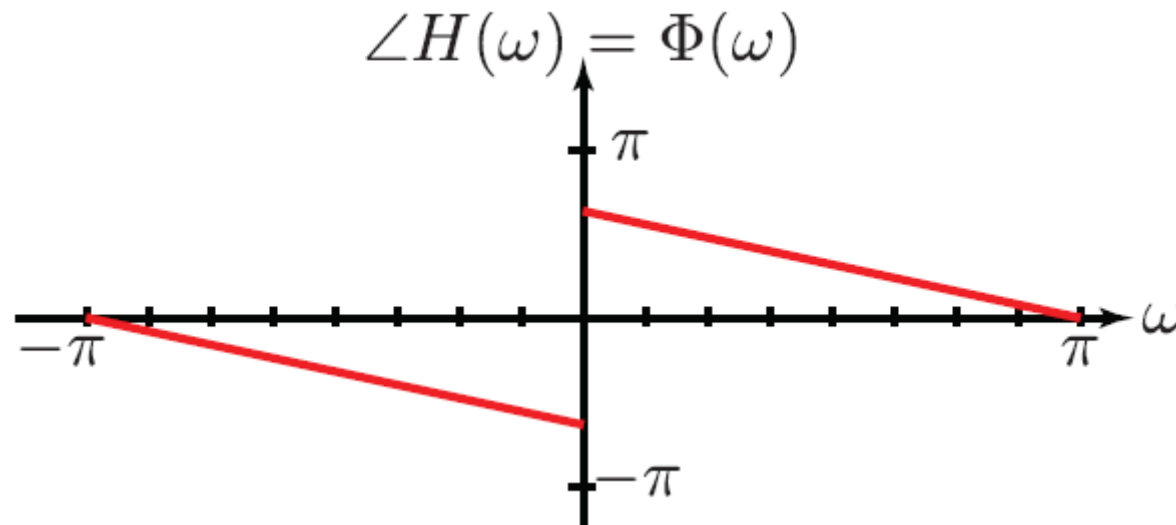
- **Phase wrapping** may occur, but the phase is still considered to be linear.



Linear Phase

Example: linear phase (high pass system)

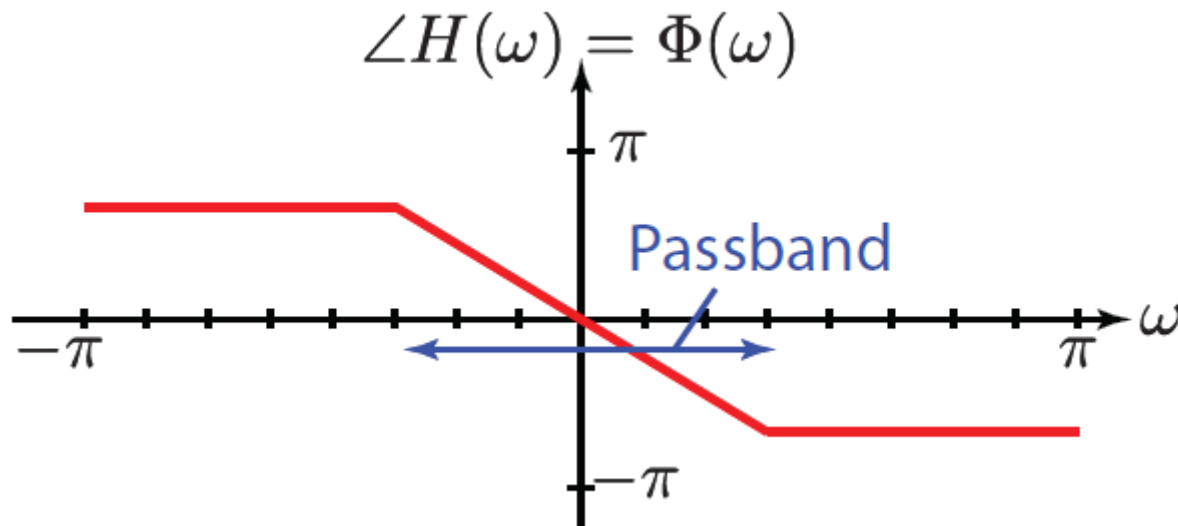
- Discontinuities at the origin still correspond to a linear phase system.



Linear Phase

Example: linear phase (low pass system)

- Linear characteristics only need to pertain to the **passband** frequencies only.



DTFT Theorems and Properties

Recall,

Property	Time Domain	Frequency Domain
Notation:	$x(n)$ $x_1(n)$ $x_2(n)$	$X(\omega)$ $X_1(\omega)$ $X_2(\omega)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting:	$x(n - k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution:	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation:	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$ $= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ real]
Wiener-Khintchine:	$r_{xx}(l) = x(l) * x(-l)$	$S_{xx}(\omega) = X(\omega) ^2$

among others ...

Group Delay

Therefore,

$$y(n) = x(n - n_0) \overset{\mathcal{F}}{\leftrightarrow} Y(\omega) = X(\omega)e^{-j\omega n_0}$$

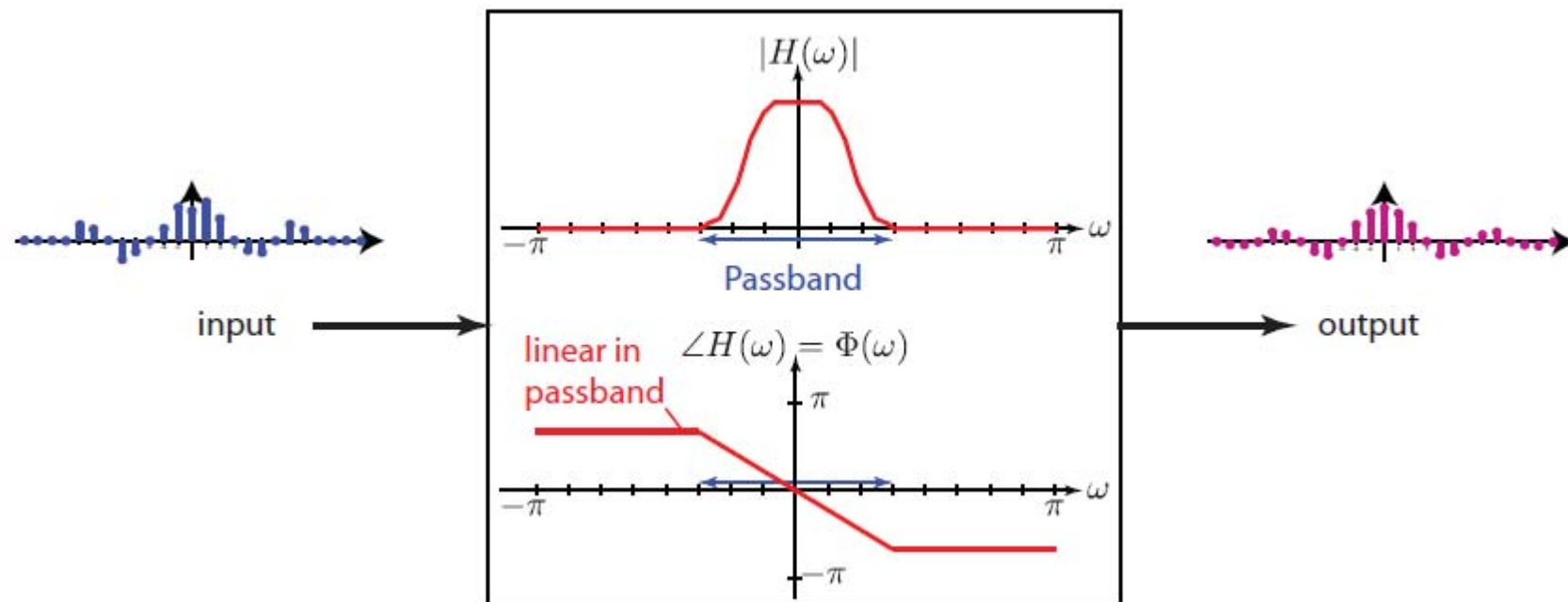
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = e^{-j\omega n_0}$$

$$\angle H(\omega) = \phi(\omega) = -\omega n_0 = -\omega \cdot \text{group delay}$$

In general (even for nonlinear phase systems),

$$\text{group delay} \equiv -\frac{d\phi(\omega)}{d\omega}$$

- Linear phase filters maintain the **relative positioning** of the sinusoids in the filter passband.
- This maintains the **structure** of the signal while removing unwanted frequency components.



Signal Magnitude versus Signal Phase

Q: Why is linear phase important?

Q: What can happen when there is loss of phase information?



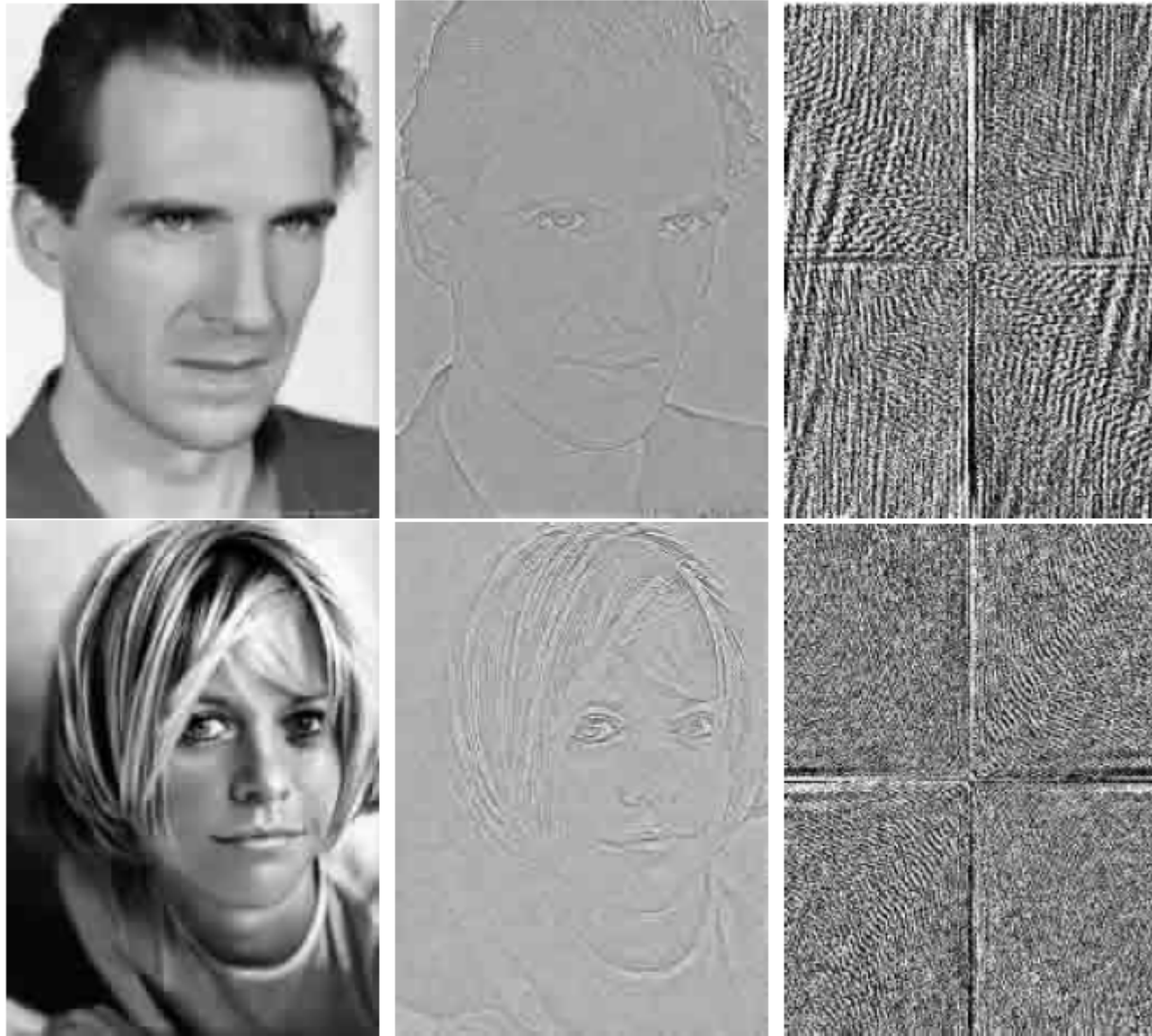
ECE 487 Real-time DSP: Practical Frequency-Selective
Digital Filter Design



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Signal Magnitude versus Signal Phase

A: To maintain the original "structure" of a signal in the passband frequency range, linear phase (or close to linear phase) is required.

Linear Phase FIR Filters

- As mentioned previously, FIR filters with the following symmetry are linear phase:

$$h(n) = \pm h(M - 1 - n) \quad n = 0, 1, 2, \dots, M - 1$$

- Note that this means that

$$h(n) = +h(M - 1 - n)$$

for $n = 0, 1, 2, \dots, M - 1$, or

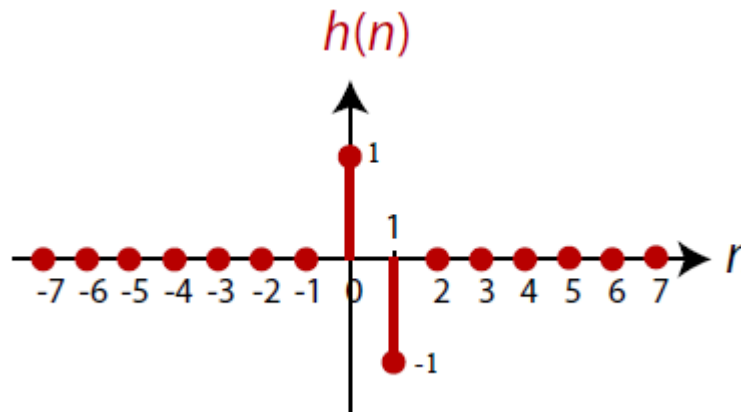
$$h(n) = -h(M - 1 - n)$$

for $n = 0, 1, 2, \dots, M - 1$

Linear Phase FIR Filters: Example

Q: Show that $h(n) = \delta(n) - \delta(n - 1)$ is linear phase by determining the associated phase and group delay.

Note: $M = 2$ and $h(n) = -h(1 - n) = -h(M - 1 - n)$ for $n = 0, 1$.



For $n = 0$, $h(0) = -h(1 - 0) = 1$ and $n = 1$, $h(1) = -h(1 - 1) = -1$

Linear Phase FIR Filters: Example

Q: Show that $h(n) = \delta(n) - \delta(n - 1)$ is linear phase by determining the associated phase and group delay.

Note: This system corresponds to:

$$\begin{aligned} y(n) &= x(n) * h(n) = x(n) * [\delta(n) - \delta(n - 1)] \\ &= x(n) * \delta(n) - x(n) * \delta(n - 1) \\ &= x(n) - x(n - 1) \quad (\text{first difference system}) \end{aligned}$$

first difference $\Leftrightarrow$ dst – time derivative $\Rightarrow$ highpass filter

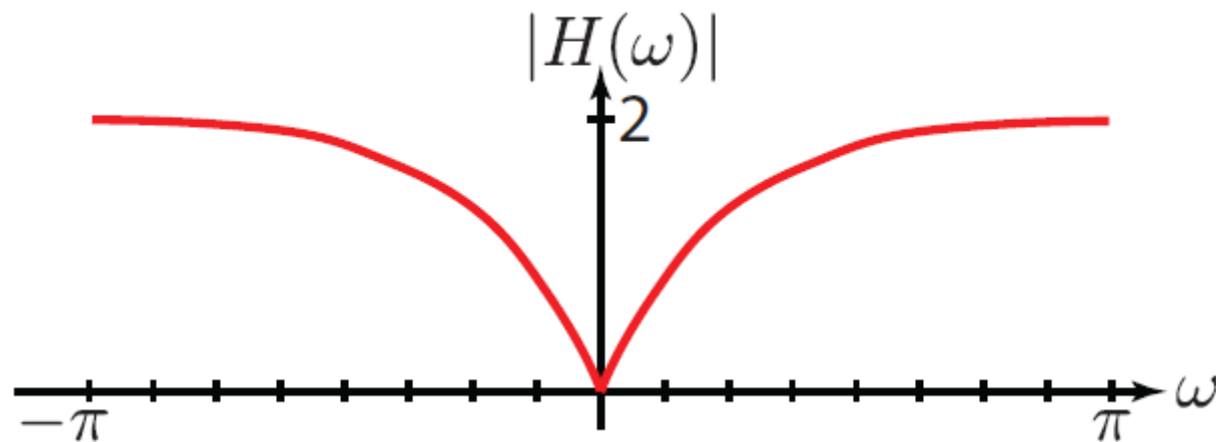
Linear Phase FIR Filters: Example

$$\begin{aligned} H(\omega) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \\ &= 1 \cdot e^{-j\omega 0} + (-1) \cdot e^{-j\omega 1} \\ &= 1 - e^{-j\omega} = e^{-j\omega/2} (e^{j\omega/2} - e^{-j\omega/2}) \\ &= e^{-j\omega/2} \cdot 2j \sin(\omega/2) = 2je^{-j\omega/2} \sin(\omega/2) \end{aligned}$$

Linear Phase FIR Filters: Example

Note:

$$\begin{aligned}|h(\omega)| &= \left| 2je^{-j\omega/2} \sin(\omega/2) \right| \\&= |2| \cdot |j| \cdot |e^{-j\omega/2}| \cdot |\sin(\omega/2)| \\&= 2 \cdot 1 \cdot 1 \cdot |\sin(\omega/2)| = 2|\sin(\omega/2)|\end{aligned}$$



Linear Phase FIR Filters: Example

$$\phi(\omega) = \angle 2je^{-j\omega/2} \sin(\omega/2)$$

$$= \underbrace{\angle 2}_{=0} + \underbrace{\angle j}_{=\pi/2} + \underbrace{\angle e^{-j\omega/2}}_{=-\omega/2} + \underbrace{\angle \sin(\omega/2)}_{=\begin{cases} 0 & 0 < \omega < \pi \\ \pi & -\pi < \omega < 0 \end{cases}}$$

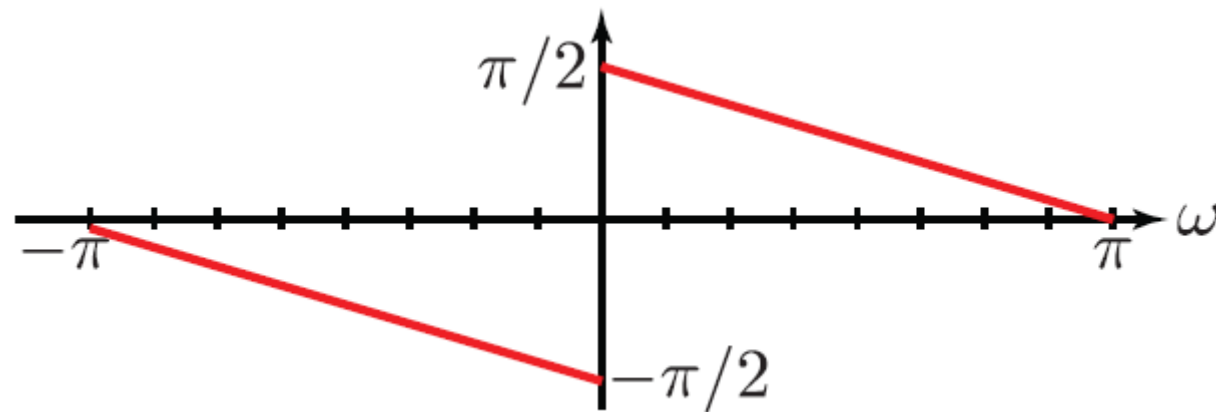
$$= \begin{cases} \frac{\pi}{2} - \frac{\omega}{2} & 0 < \omega < \pi \\ \frac{3\pi}{2} - \frac{\omega}{2} - 2\pi & -\pi < \omega < 0 \end{cases}$$

$$= \begin{cases} \frac{\pi - \omega}{2} & 0 < \omega < \pi \\ \frac{-\pi - \omega}{2} & -\pi < \omega < 0 \end{cases}$$

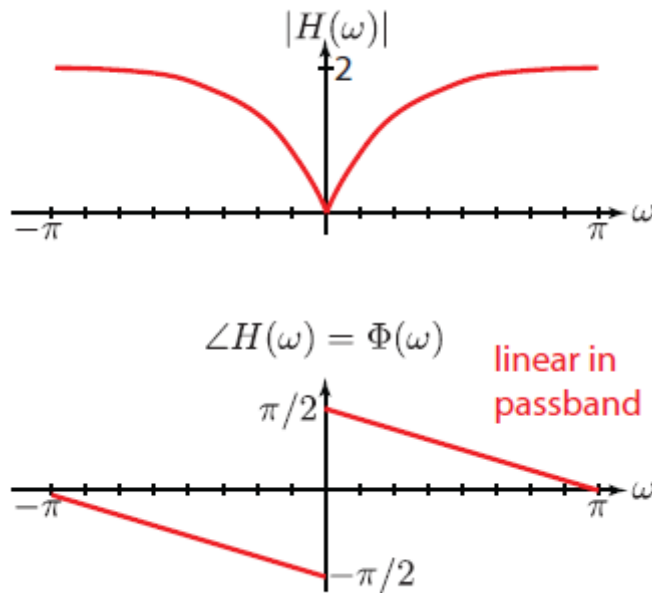
Linear Phase FIR Filters: Example

$$\phi(\omega) = \begin{cases} \frac{\pi - \omega}{2} & 0 < \omega < \pi \\ -\frac{\pi - \omega}{2} & -\pi < \omega < 0 \end{cases}$$

$$\angle H(\omega) = \Phi(\omega)$$



Linear Phase FIR Filters: Example



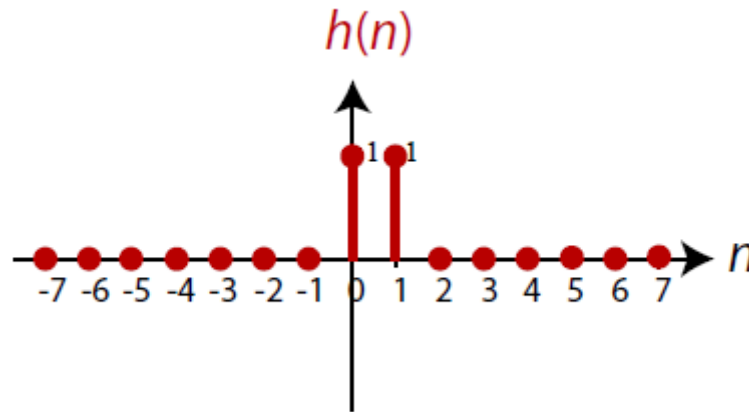
Group Delay:

$$\begin{aligned} -\frac{d\phi(\omega)}{d\omega} &= \frac{1}{2} - \pi\delta(\omega) \\ &= \begin{cases} \frac{1}{2} & \omega \neq 0 \\ -\pi\delta(\omega) & \omega = 0 \end{cases} \\ &= \frac{1}{2} = \text{constant} \\ &\quad \text{(in passband)} \end{aligned}$$

Linear Phase FIR Filters: Example 2

Q: Show that $h(n) = \delta(n) + \delta(n - 1)$ is linear phase by determining the associated phase and group delay.

Note: $M = 2$ and $h(n) = +h(1 - n) = +h(M - 1 - n)$ for $n = 0, 1$.



For $n = 0$, $h(0) = +h(1 - 0) = 1$ and $n = 1$, $h(1) = +h(1 - 1) = -1$

Linear Phase FIR Filters: Example 2

Q: Show that $h(n) = \delta(n) + \delta(n - 1)$ is linear phase by determining the associated phase and group delay.

Note: This system corresponds to:

$$\begin{aligned} y(n) &= x(n) * h(n) = x(n) * [\delta(n) + \delta(n - 1)] \\ &= x(n) * \delta(n) + x(n) * \delta(n - 1) \\ &= x(n) + x(n - 1) \quad (\text{scaled averaging system}) \end{aligned}$$

avearager $\Leftrightarrow$ dst – time smooter $\Rightarrow$ lowpass filter

Linear Phase FIR Filters: Example 2

$$\begin{aligned} H(\omega) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \\ &= 1 \cdot e^{-j\omega 0} + (+1) \cdot e^{-j\omega 1} \\ &= 1 + e^{-j\omega} = e^{-j\omega/2} (e^{j\omega/2} + e^{-j\omega/2}) \\ &= e^{-j\omega/2} \cdot 2\cos(\omega/2) = 2e^{-j\omega/2}\cos(\omega/2) \end{aligned}$$

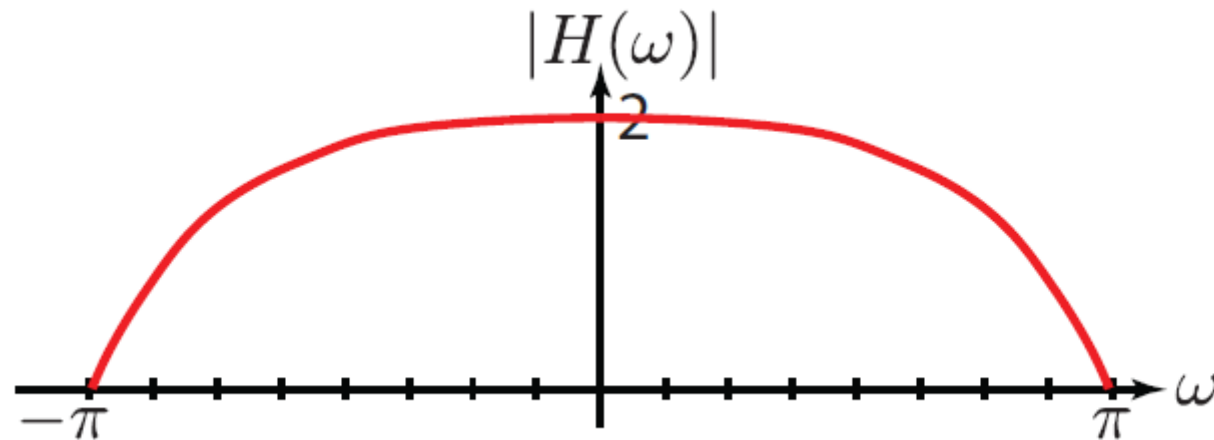
Linear Phase FIR Filters: Example 2

Note:

$$|h(\omega)| = |2e^{-j\omega/2} \cos(\omega/2)|$$

$$= |2| \cdot |e^{-j\omega/2}| \cdot |\cos(\omega/2)|$$

$$= 2 \cdot 1 \cdot |\cos(\omega/2)| = 2|\cos(\omega/2)|$$



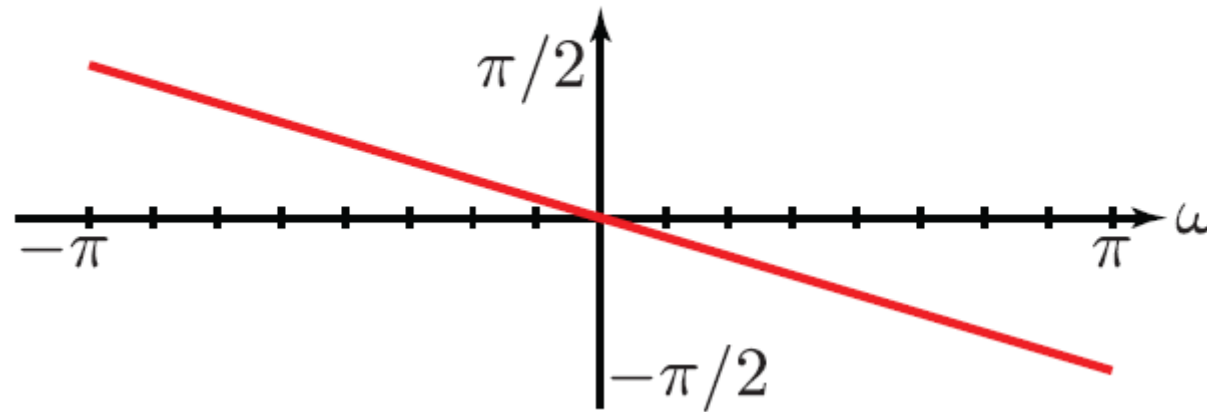
Linear Phase FIR Filters: Example 2

$$\begin{aligned}\phi(\omega) &= \angle 2e^{-j\omega/2} \cos(\omega/2) \\&= \underbrace{\angle 2}_{=0} + \underbrace{\angle e^{-j\omega/2}}_{=-\omega/2} + \underbrace{\angle \cos(\omega/2)}_{=0} \\&= -\frac{\omega}{2}\end{aligned}$$

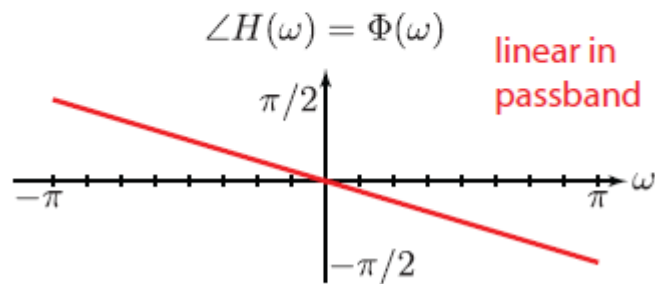
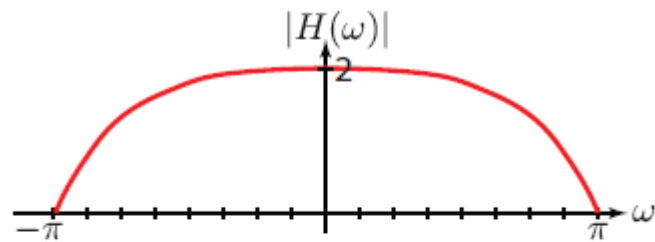
Linear Phase FIR Filters: Example 2

$$\phi(\omega) = -\frac{\omega}{2}$$

$$\angle H(\omega) = \Phi(\omega)$$



Linear Phase FIR Filters: Example 2



Group Delay:

$$-\frac{d\phi(\omega)}{d\omega} = \frac{1}{2} = \text{constant}$$

(in passband)

Ideal Filters

An **ideal lowpass filter** is given by:

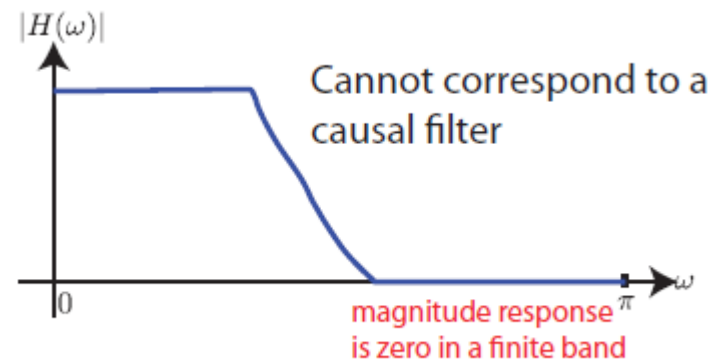
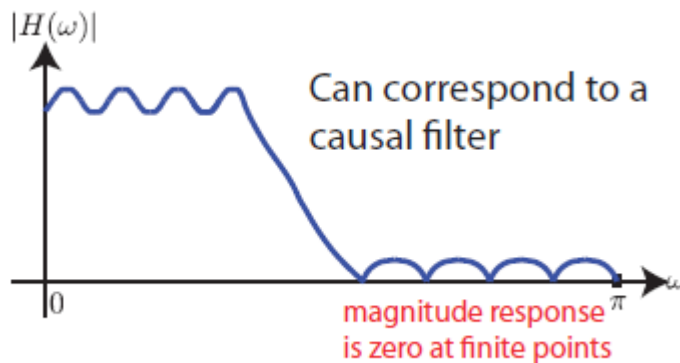
$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

The impulse response is given by:

$$h(n) = \begin{cases} \frac{\omega_c}{\pi} & n = 0 \\ \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n} & n \neq 0 \end{cases}$$

Limitations of Practical Filters

- An ideal filter is **not causal** since $h(n) \neq 0$ for $n < 0$.
- From the Paley-Wiener Theorem: for causal LTI systems where necessarily $h(n) = 0$ for $n < 0$, $|H(\omega)|$ can be zero **only** at a **finite set of points** in a frequency interval, but not over a finite band of frequencies.



Limitations of Practical Filters

- **Rippling** occur in the passband and stopband -Why?
- imposing causality is like truncating $h(n)$ so it has no negative part, which results in **Gibbs phenomenon** - i.e., ringing/rippling effect for $H(\omega)$

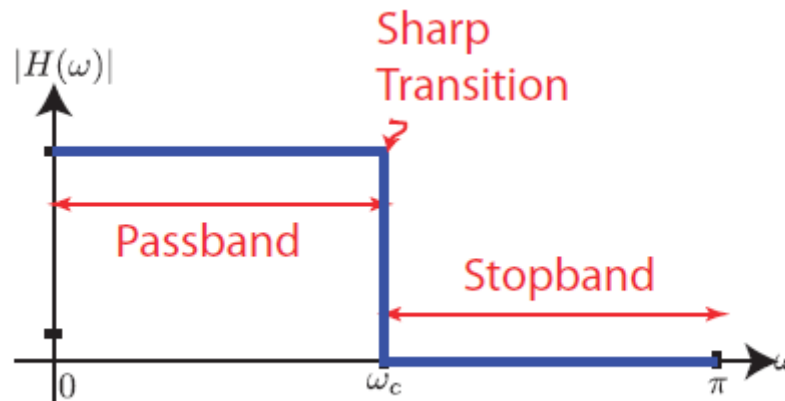
ringing $\overset{\mathcal{F}}{\leftrightarrow}$ truncation (Gibbs in time domain)

truncation $\overset{\mathcal{F}}{\leftrightarrow}$ ringing ($H(\omega)$ pass/stopband rippling)

- In addition, filters with finite parameters will demonstrate a measurable transition between passband and stopband.

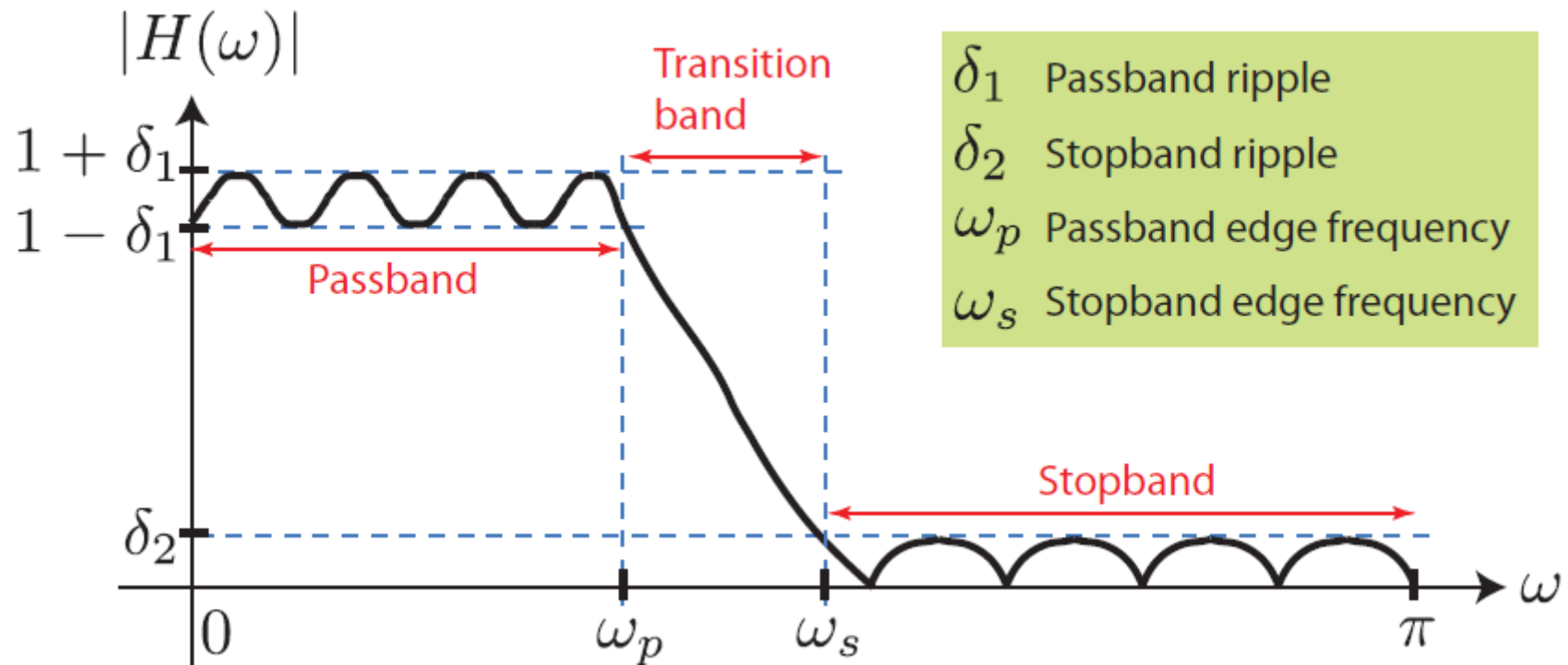
Practical Frequency Selective Filters

- Ideal filter characteristics of sharp transitions and at gains may not be absolutely necessary for most practical applications.



- Relaxing these conditions provides an opportunity to realize causal finite parameter filters that approximate ideal filters as close as we desire.

Practical Frequency Selective Filters



Desired Frequency Response

Given: $H_d(\omega)$ (**desired** frequency response)

$$H_d(\omega) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

Recall for a digital **FIR** implementation, $h_d(n)$ needs to be **finite** duration; say, of length **M**. Therefore, it is required that $h_d(n) = 0$ for $n < 0$ and $n > M - 1$.

In general, $h_d(n)$ is infinite duration . . .

Design of Linear-Phase FIR Filters using Windows

Q: How do we make $h_d(n)$ finite duration?

A: windowing . . .

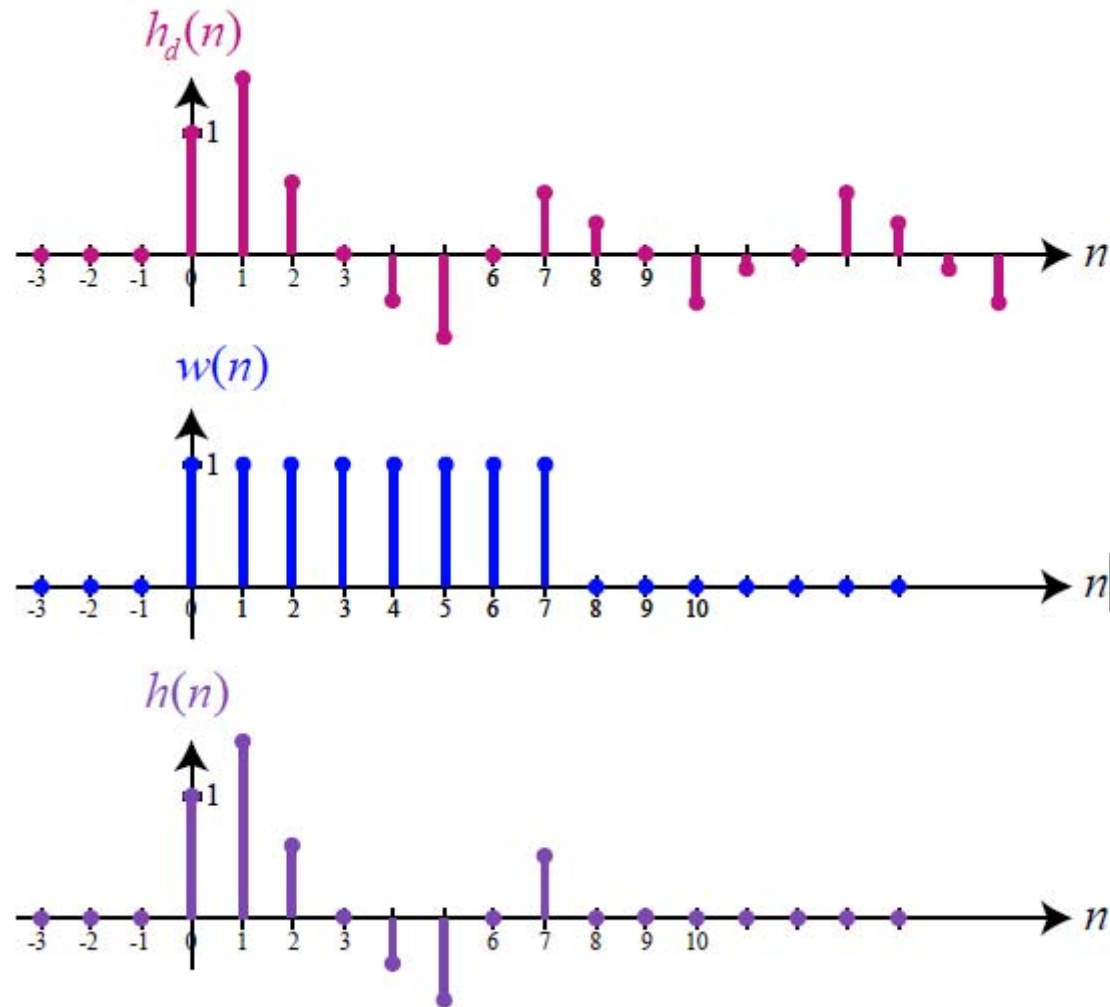
Consider the rectangular window

$$w(n) = \begin{cases} 1 & n = 0, 1, \dots, M-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} h(n) &= h_d(n) w(n) \\ &= \begin{cases} h_d(n) & n = 0, 1, \dots, M-1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Example

Rectangular window, $M = 8$



Windowing Distortion

Q: What is the distortion introduced by windowing?

A: Look in the frequency domain . . .

convolution $\xleftrightarrow{\mathcal{F}}$ multiplication

multiplication $\xleftrightarrow{\mathcal{F}}$ convolution

$$h_d(n) w(n) \xleftrightarrow{\mathcal{F}} H_d(\omega) * W(\omega)$$

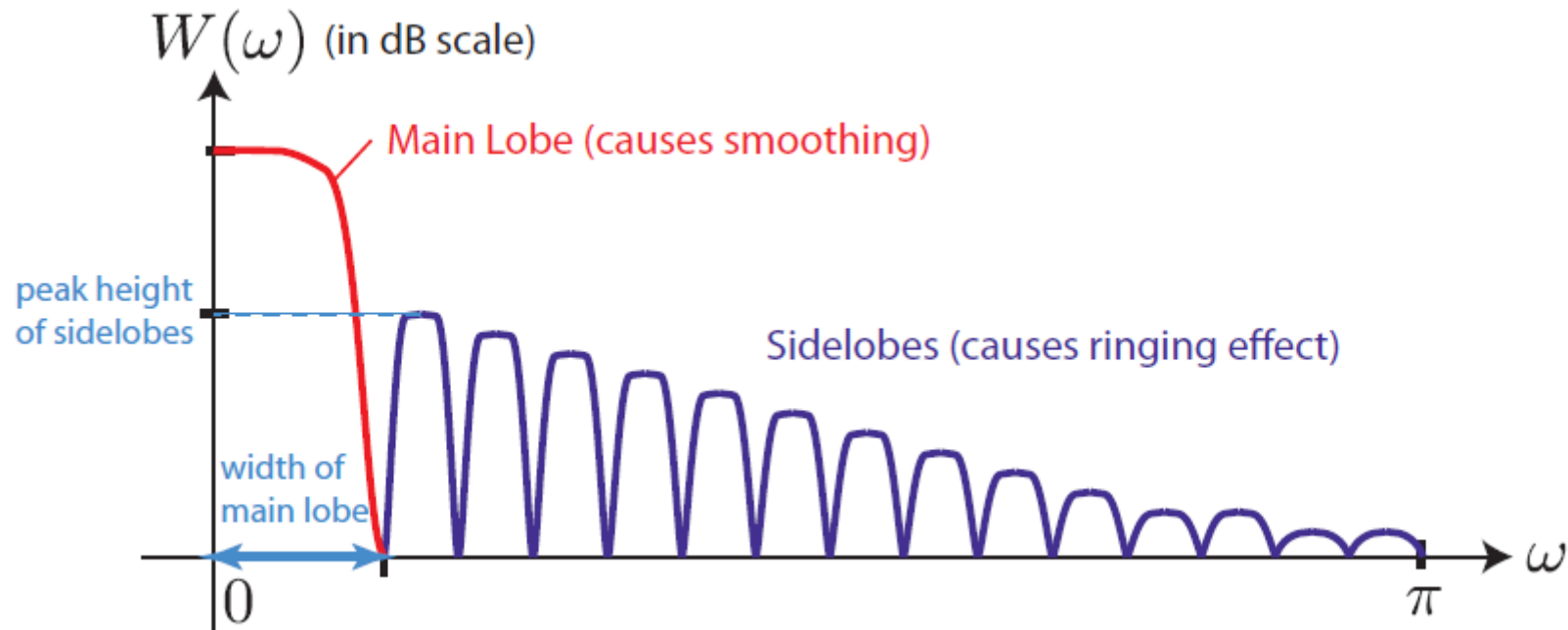
$$h(n) = h_d(n) w(n) \xleftrightarrow{\mathcal{F}} H(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{H_d(\omega) W(\omega - v)}_{\text{depends on } w(n)} dv$$

$$W(\omega) = \sum_{n=0}^{M-1} w(n) e^{-j\omega n}$$

Windowing Distortion

- Convolution of $H_d(\omega)$ with $W(\omega)$ has the effect of smoothing out the frequency response of the resulting filter.
- For no distortion from windowing, want $W(\omega)$ to be close to a delta function, $\delta(\omega)$
- $W(\omega)$ is partially characterized by:
 - main lobe width (in rad/s)
 - peak amplitude of side lobe (in dB)

Windowing Distortion



- increasing window length generally reduces the width of the main lobe
- peak of sidelobes is generally independent of M

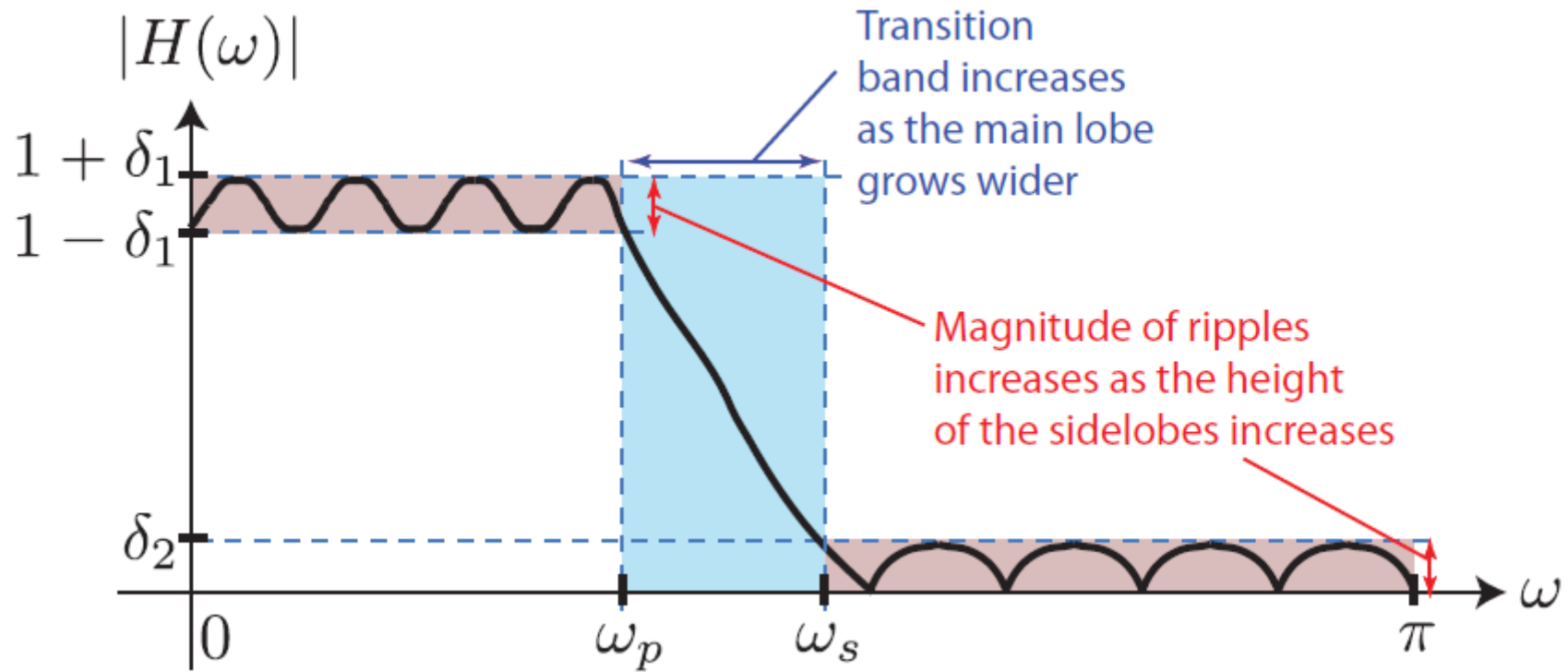
Characteristics of Different Windows

Window type	Main lobe width	Peak sidelobe (dB)
Rectangular	$4\pi / M$	-13
Bartlett	$8\pi / M$	-25
Hanning	$8\pi / M$	-31
Hamming	$8\pi / M$	-41
Blackman	$12\pi / M$	-57

Note:

- the larger the main lobe, the larger the filter transition region
- the larger the peak sidelobe, the higher the degree of ringing in the pass/stopbands

Effects of Windowing in Frequency Domain



Design of Linear-Phase FIR Filters using Windows

1. Begin with a desired frequency response $H_d(\omega)$ that is linear phase with a delay of $(M - 1)/2$ units in anticipation of forcing the filter to be length M .

Example:

$$H_d(\omega) = \begin{cases} 1 \cdot e^{-j\omega(M-1)/2} & 0 \leq |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Design of Linear-Phase FIR Filters using Windows

2. The corresponding impulse response is given by:

$$h_d(n) = \frac{1}{2\pi} \int_{-w_c}^{w_c} H_d(\omega) e^{j\omega n} d\omega$$

Example:

$$h_d(n) = \frac{1}{2\pi} \int_{-w_c}^{w_c} e^{j\omega(n-(M-1)/2)} d\omega$$

$$= \begin{cases} \frac{\sin \omega_c (n - \frac{M-1}{2})}{\pi (n - \frac{M-1}{2})} & n \neq \frac{M-1}{2} \\ \frac{\omega_c}{\pi} & n = \frac{M-1}{2} \end{cases} \quad \text{(if M is odd)}$$
$$= \begin{cases} \frac{\sin \omega_c (n - \frac{M-1}{2})}{\pi (n - \frac{M-1}{2})} & \text{(if M is even)} \end{cases}$$

Design of Linear-Phase FIR Filters using Windows

3. Multiply $h_d(n)$ with a window of length M .

$$h(n) = h_d(n) \cdot w(n)$$

Example: rectangular window

$$w(n) = \begin{cases} 1 & n = 0, 1, \dots, M-1 \\ 0 & \text{otherwise} \end{cases}$$

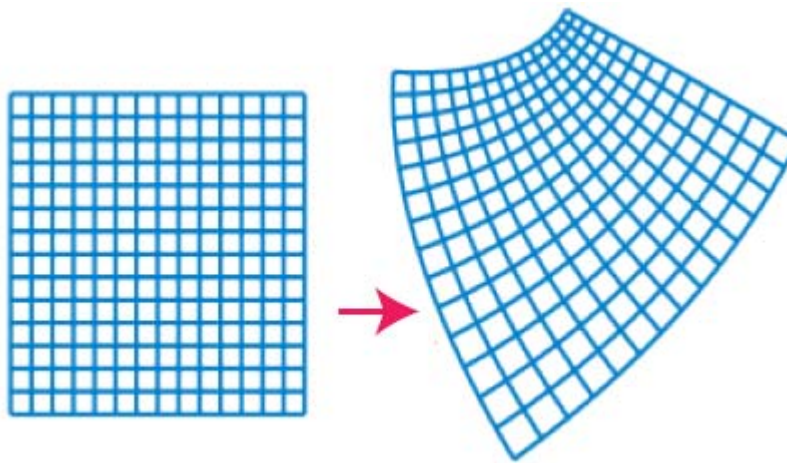
$$h(n) = h_d(n) \cdot w(n)$$
$$= \begin{cases} \frac{\sin \omega_c (n - \frac{M-1}{2})}{\pi (n - \frac{M-1}{2})} & n \neq \frac{M-1}{2} \\ \frac{\omega_c}{\pi} & n = \frac{M-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Can we get better filter performance?

- Yes. Use IIR filters.
- IIR digital filters can be designed by converting a well-known analog filter into a digital one.
- For the same number of parameters, better compromises between ringing and transition band width can be found.

IIR Filter Design via Bilinear Transformation

- bilinear transformation: mapping from the s-plane to the z-plane
 - **conformal mapping** (mapping that preserves local angles among curves) that transforms the **vertical axis of the s-plane** into the **unit circle in the z-plane**

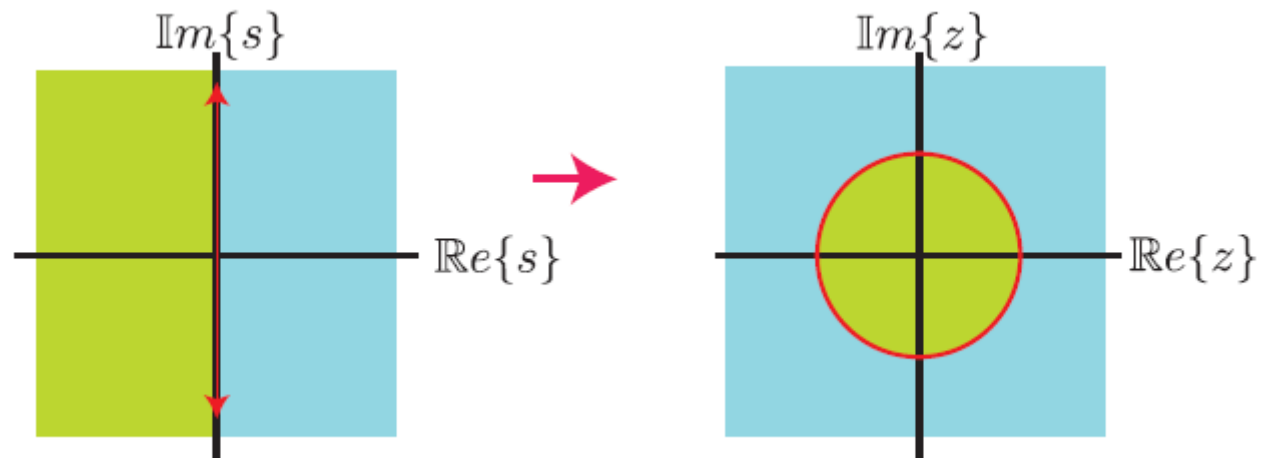


IIR Filter Design via Bilinear Transformation

- bilinear transformation: mapping from the s-plane to the z-plane
 - **conformal mapping** (mapping that preserves local angles among curves) that transforms the **vertical axis of the s-plane** into the **unit circle in the z-plane**
 - all points in the left half plane (LHP) of s are mapped into corresponding points inside the unit circle in the z-plane
 - all points in the right half plane (RHP) of s are mapped into corresponding points outside the unit circle in the z-plane

Digital Filtering in Time Domain

- bilinear transformation: mapping from the s-plane to the z-plane



Bilinear Transformation: Example

$$H_a(s) = \frac{Y(s)}{X(s)} = \frac{b}{s + a}$$

$$Y(s)(s + a) = bX(s)$$

$$sY(s) + aY(s) = bX(s)$$

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Note: we will use $\frac{dy(t)}{dt}$ and $y'(t)$ interchangeably.

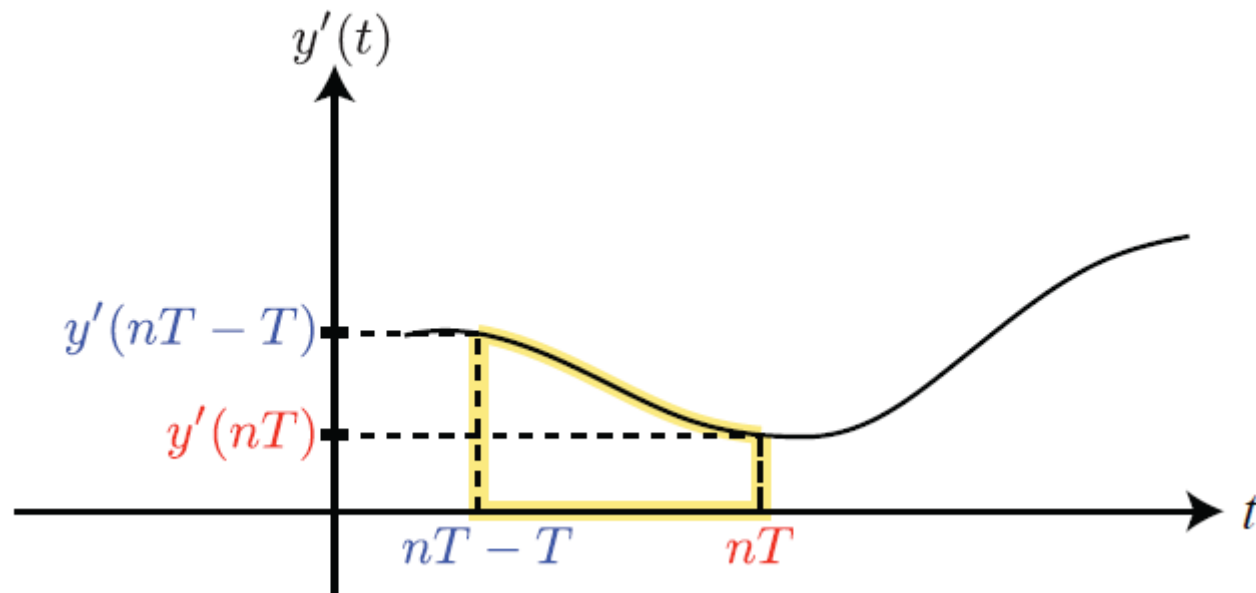
Bilinear Transformation: Example

Consider:

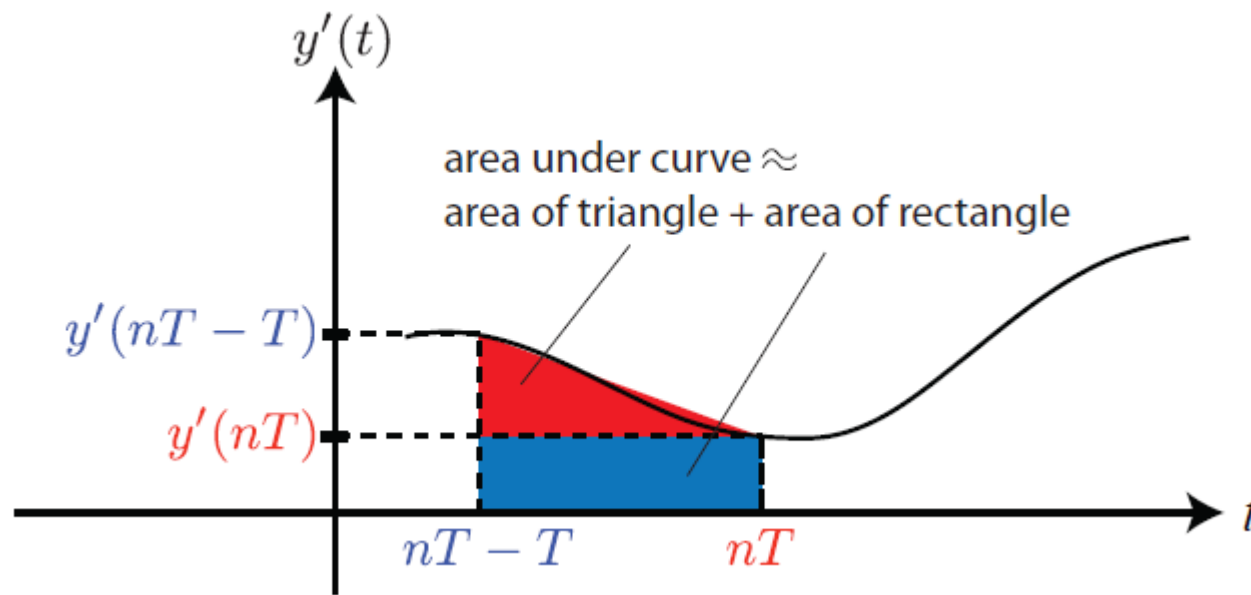
$$y(\textcolor{red}{t}) = \underbrace{\int_{\textcolor{blue}{t}_0}^{\textcolor{red}{t}} y'(\tau) d\tau}_{\equiv I} + y(\textcolor{blue}{t}_0)$$

Let $\textcolor{red}{t} = \textcolor{red}{nT}$ and $\textcolor{blue}{t}_0 = \textcolor{blue}{nT} - T$ and using the trapezoidal approximation of the integral:

$$y(\textcolor{red}{nT}) = \underbrace{\frac{T}{2} [y'(\textcolor{red}{nT}) + y'(\textcolor{blue}{nT} - T)]}_{\text{we will show this } \approx I} + y(\textcolor{blue}{nT} - T)$$



$$\text{Area under curve} = I = \int_{nT-T}^{nT} y'(\tau) d\tau$$



$$\begin{aligned}
 I &= \int_{nT-T}^{nT} y'(\tau) d\tau \approx b_{rect} \times h_{rect} + \frac{b_{tri} \times h_{tri}}{2} \\
 &= T \cdot y'(nT) + \frac{T \cdot (y'(nT - T) - y'(nT))}{2} \\
 &= \frac{T}{2} [y'(nT) + y'(nT - T)]
 \end{aligned}$$

Bilinear Transformation: Example

Therefore we indeed have:

$$y(nT) = \frac{T}{2} [y'(nT) + y'(nT - T)] + y(nT - T)$$

Plugging $t = nT, nT - T$ into $y'(t) + ay(t) = bx(t)$ gives:

$$y(nT) = \frac{T}{2} \left[\underbrace{-ay(nT) + bx(nT)}_{y'(nT)} + \underbrace{-ay(nT - T) + bx(nT - T)}_{y'(nT - T)} \right] + y(nT - T)$$

and letting $x(n) \equiv x(nT)$ and $y(n) \equiv y(nT)$, we obtain . . .

Bilinear Transformation: Example

$$y(n) = \frac{T}{2} [-ay(n) + bx(n) - ay(n-1) + bx(n-1)] + y(n-1)$$

$$\left(1 + \frac{aT}{2}\right)y(n) - \left(1 - \frac{aT}{2}\right)y(n-1) = \frac{bT}{2}[x(n) + x(n-1)]$$

$$\left(1 + \frac{aT}{2}\right)Y(z) - \left(1 - \frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}[X(z) + z^{-1}X(z)]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

Bilinear Transformation: Example

Compare:

$$H(z) = \frac{b}{\frac{2}{T} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) + a}$$

to

$$H_a(s) = \frac{b}{s + a}$$

Therefore, bilinear transformation mapping is:

$$s = \frac{2}{T} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right)$$

Bilinear Transformation

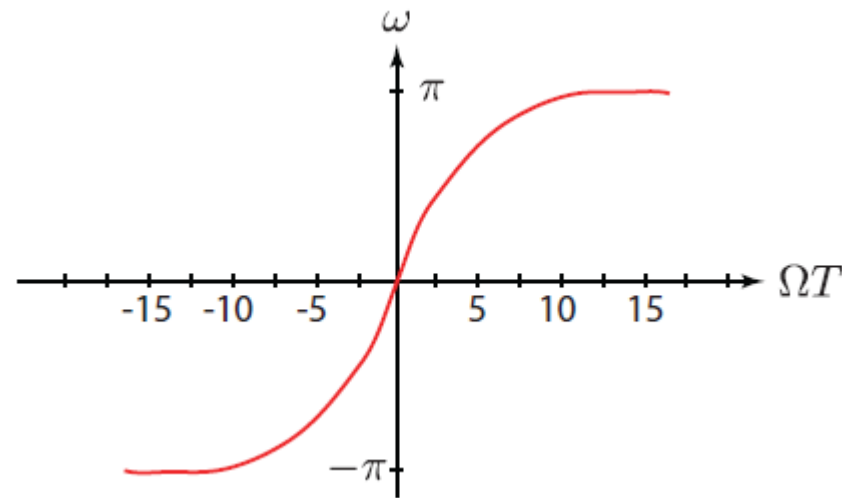
The mapping $s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)$ will work for any order of differential equation to convert $H_a(s)$ to $H(z)$.

General Methodology:

1. Start with $H_a(s)$ expression.
2. Determine T through the problem specifications.
3. $H(z) = H_a \left(\frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right) \right)$

Bilinear Transformation

For $s = j\Omega$ and $z = e^{j\omega}$:



The entire $-\infty < \Omega < \infty$ axis is mapped to $-\pi < \omega < \pi$. There is a huge compression of the frequency response at large Ω -values.

Bilinear Transformation

For $s = j\Omega$ and $z = e^{j\omega}$:

